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GLOBAL ATTRACTOR FOR QUASILINEAR PARABOLIC SYSTEMS INVOLVING WEIGHTED *p*-LAPLACIAN OPERATORS

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Abstract

This paper is concerned with the long-time behaviour of solutions for a class of quasilinear parabolic systems. In particular, we prove the existence of the compact global attractor in $L^2(\Omega) \times L^2(\Omega)$ for the multi-valued semiflow $\mathcal{G}$.

1. Introduction

In this paper, we are concerned with the existence of the global attractor of the solutions for a class of nonlinear parabolic systems involving weighted p-Laplacian operators of the type $(\mathcal{S})$

$$\frac{\partial u_1}{\partial t} + \mathcal{A}_1 u_1 + a_1(x)|u_1|^{q_1 - 2} u_1 = f_1(x, u_1, u_2) \quad \text{in } \Omega \times \mathbb{R}^+, \tag{1}$$

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$$\frac{\partial u_2}{\partial t} + \mathcal{A}_2 u_2 + a_2(x)|u_2|^{q_2-2}u_2 = f_2(x, u_1, u_2) \quad \text{in } \Omega \times \mathbb{R}^+, \tag{2}$$

$$u_1 = u_2 = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+, \tag{3}$$

$$(u_1(x, 0), u_2(x, 0)) = (\varphi_1(x), \varphi_2(x)) \quad \text{in } \Omega, \tag{4}$$

where

$$\mathcal{A}_i u = -\operatorname{div}\left(\sigma_i(x)|\nabla u|^{p_i-2}\nabla u\right), \quad i = 1, 2,$$

and $2 \leq p_i < q_i$, $p_i < N$, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and the functions σ_i, f_i, $i = 1, 2$, satisfy some conditions specified later.

This systems contains some important classes of parabolic problems. When $\sigma_i = \text{const.} > 0$, $p_i = 2$, then $(\mathcal{S})$ becomes the semilinear heat systems

$$(\mathcal{S}1)\begin{cases} \dfrac{\partial u_1}{\partial t} - c_1\Delta u_1 = f_1(x, u_1, u_2) & \text{in } \Omega \times \mathbb{R}^+, \\ \dfrac{\partial u_2}{\partial t} - c_2\Delta u_2 = f_2(x, u_1, u_2) & \text{in } \Omega \times \mathbb{R}^+, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ (u_1(x, 0), u_2(x, 0)) = (\varphi_1(x), \varphi_2(x)) & \text{in } \Omega, \end{cases}$$

and when $p_i \neq 2$, then $(\mathcal{S})$ becomes the p-Laplacian systems

$$(\mathcal{S}2)\begin{cases} \dfrac{\partial u_1}{\partial t} - c_1\Delta_{p_1} u_1 = f_1(x, u_1, u_2) & \text{in } \Omega \times \mathbb{R}^+, \\ \dfrac{\partial u_2}{\partial t} - c_2\Delta_{p_2} u_2 = f_2(x, u_1, u_2) & \text{in } \Omega \times \mathbb{R}^+, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ (u_1(x, 0), u_2(x, 0)) = (\varphi_1(x), \varphi_2(x)) & \text{in } \Omega. \end{cases}$$

Systems $(\mathcal{S}1)$ have received extensive investigations in the past several decades, see, e.g., [20] and references therein. The authors always use the nice properties of Δ, to obtain the existence and long-time behaviour of solutions to systems $(\mathcal{S}1)$.

Systems $(\mathcal{S}2)$ appear in the study of non-Newtonian fluids and non-Newtonian filtration, see [3, 13]. The quantity (p_1, p_2) is the characteristic of the medium. Media with $(p_1, p_2) > (2, 2)$ are called *dilatant fluids* and those with $(p_1, p_2) < (2, 2)$ are called *pseudoplastics*. If $(p_1, p_2) = (2, 2)$, then they are Newtonian fluids. Many authors have contributed for a better understanding of several questions related to $(\mathcal{S}2)$, for example, regularity, existence, asymptotic behaviour, and global attractor of a solution. The basic tools that have been used are a priori estimates, degree theory, and the super-subsolution method, see, for example, [5, 6, 9, 11, 12, 20].

Parabolic systems of (p_1, p_2)-Laplacian type arise in many application and the more interesting question concerning these systems is to understand the asymptotic behaviour of solutions when time goes to infinity. The study of the asymptotic behaviour of the system is giving us relevant information about the structure of the phenomenon described in the model.

Recently, Anh and Hung [2] discussed the existence and long-time behaviour of solutions of the quasilinear parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}(\sigma(x)|\nabla u|^{p-2}|\nabla u|) + f(u) = g(u) & \text{in } \Omega \times \mathbb{R}^+, \\ u_{|t=0} = u_0(x) & \text{in } \Omega, \\ u_{|\partial\Omega} = 0, & \end{cases}$$

via the concept of global attractors for multi-valued semiflows and without uniqueness. The authors use the compactness method and monotonicity method [18, Chapters 1-2] and use the theory of global attractor for multi-valued semiflows of Melnik and Valero [21].

In this paper, motivated by the ideas in [2], we generalize and extend the results of [2] to systems $(\mathcal{S})$. In this way, we obtain a result and extend some known results related to the p-Laplacian.

The outline of the paper is as follows: In Section 2, we make some assumptions and prove the existence of the solutions. Section 3 is devoted to the m-semiflow generated by the systems $(\mathcal{S})$.

2. Existence

2.1. Notations and assumptions

Let Ω be a smooth and bounded domain in $\mathbb{R}^N (N \geq 2)$. Set for $t > 0$, $Q_t := \Omega \times (0, t)$, $S_t := \partial\Omega \times (0, t)$.

We represent the weighted Sobolev space $\mathcal{D}_0^{1,p}(\Omega, \sigma)$ defined as the closure of $\mathcal{C}_0^\infty(\Omega)$ in the norm

$$\|u\|_{\mathcal{D}_0^{1,p}(\Omega, \sigma)} = \left(\int_\Omega \sigma |\nabla u|^p \right)^{1/p}.$$

Let $\mathcal{D}^{-1,p^*}(\Omega, \sigma)$ be the dual space of $\mathcal{D}_0^{1,p}(\Omega, \sigma)$, where p^* is the conjugate of p, i.e., $\frac{1}{p} + \frac{1}{p^*} = 1$.

We denote

$$V_i = L^{p_i}(0, T; \mathcal{D}_0^{1,p_i}(\Omega, \sigma_i)) \cap L^{q_i}(Q_T) \cap L^2(Q_T), \qquad i = 1, 2,$$

$$V_i^* = L^{p_i^*}(0, T; \mathcal{D}^{-1,p_i^*}(\Omega, \sigma_i)) + L^{q_i^*}(Q_T) + L^2(Q_T), \quad i = 1, 2.$$

The operator $\mathcal{A}_i$ is such that for $i = 1, 2$,

$$\mathcal{A}_i : \mathcal{D}_0^{1,p_i}(\Omega, \sigma_i) \to \mathcal{D}^{-1,p'}(\Omega, \sigma_i),$$

$$u \mapsto \mathcal{A}_i u = -\operatorname{div}(\sigma_i(x)|\nabla u|^{p_i - 2}\nabla u),$$

and it satisfies the following properties [2], which are easily proved by using similar arguments as in [18, Chapter 2]:

(1) $\mathcal{A}_i$ is monotonic, that is, $\langle \mathcal{A}_i u - \mathcal{A}_i v, u - v \rangle \geq 0, \quad \forall u, v \in \mathcal{D}_0^{1, p_i}$ (Ω, σ);

(2) $\mathcal{A}_i$ is hemicontinuous, that is, for each $u, v, w \in \mathcal{D}_0^{1, p_i}(\Omega, \sigma_i)$, the function $\lambda \mapsto \langle \mathcal{A}_i(u + \lambda v), w \rangle$ is continuous from $\mathbb{R}$ to $\mathbb{R}$;

(3) Assume that

$$\begin{cases} u_n \to u \text{ in } L^{p_i}(0, T; \mathcal{D}_0^{1, p_i}(\Omega, \sigma_i)), \\ \mathcal{A}_i u_n \to \Psi_i \text{ in } L^{p_i'}(0, T; \mathcal{D}^{-1, p_i}(\Omega, \sigma_i)) \text{ and } \overline{\lim_{n \to \infty}} \langle \mathcal{A}_i u_n, u_n \rangle \leq \langle \Psi_i, u \rangle, \end{cases}$$

then $\Psi_i = \mathcal{A}_i u$.

In the sequel, the same symbol c will be used to indicate some positive constants, possibly different from each other, appearing in the various hypotheses and computations and depending only on data. When we need to fix the precise value of one constant, then we shall use a notation like M_i, $i = 1, 2, \ldots,$ instead.

In the sequel, we shall present the following assumptions:

$$\text{(H1)} \begin{cases} \text{The function } \sigma_i : \Omega \to \mathbb{R} \text{ satisfies the following:} \\ \qquad \sigma_i \in L^1_{loc}(\Omega) \text{ and for some } \alpha_i \in (0, p_i), \\ \liminf_{x \to z} |x - z|^{-\alpha_i} \sigma_i(x) > 0, \text{ for all } z \in \overline{\Omega}, \ (i = 1, 2). \end{cases}$$

$$\text{(H2)} \begin{cases} f_i \in C^1(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}) \text{ and } f_i(x, u_1, u_2) \text{ satisfies:} \\ \displaystyle\sum_{i=1}^{2} f_i(x, u_1, u_2) u_i \leq M \sum_{i=1}^{2} |u_i|^{p_i} + h(x), \\ \text{where } M \text{ is a positive constant; } h \in L^\infty(\Omega). \end{cases}$$

$$\text{(H3)} \begin{cases} \text{The function } a_i : \Omega \to \mathbb{R} \text{ satisfies the following:} \\ \sigma_i \in L(^\infty \Omega) \text{ and } a_i(x) \geq d_i \geq d > 0, \quad (i = 1, 2). \end{cases}$$

Before proving the existence result, we need two auxiliary lemma.

Lemma 2.1 (Ghidaghia lemma [25]). *Let y be a positive absolutely continuous function on $(0, \infty)$, which satisfies*

$$y' + \mu y^{q+1} \leq \lambda,$$

with $q > 0$, $\mu > 0$, and $\lambda \geq 0$. Then for $t > 0$,

$$y(t) \leq \left(\frac{\lambda}{\mu}\right)^{\frac{1}{q+1}} + [\mu q t]^{\frac{-1}{q}}.$$

Lemma 2.2. (i) *If $u_i \in V_i$ and $\frac{du_i}{dt} \in V_i^*$, then $u_i \in C([0, T]; L^2(\Omega))$;*

(ii) *Let $\{u_n\}$ be a bounded sequence in $L^{p_i}(0, T; \mathcal{D}_0^{1, p_i}(\Omega, \sigma_i))$. Then $\{u_n\}$ converges almost everywhere in Q_T up to a subsequence.*

Proof. (i) We select a sequence $u_i^n \in C([0, T]; \mathcal{D}_0^{1, p_i}(\Omega, \sigma_i) \cap L^{q_i}(\Omega) \cap L^2(\Omega))$, such that

$$u_i^n \to u_i \quad \text{in} \quad V_i,$$

$$\frac{\partial u_i^n}{\partial t} \to \frac{\partial u_i}{\partial t} \quad \text{in} \quad V_i^*.$$

Then, for all $t, s \in [0, T]$, we have

$$\|u_i^n(t) - u_i^m(t)\|_{L^2(\Omega)}^2 = \|u_i^n(s) - u_i^m(s)\|_{L^2(\Omega)}^2$$

$$+ 2\int_s^t \left\langle \frac{du_i^n(\tau)}{dt} - \frac{du_i^m(\tau)}{dt}, u_i^n(\tau) - u_i^m(\tau) \right\rangle d\tau.$$

We choose s so that

$$\|u_i^n(s) - u_i^m(s)\|_{L^2(\Omega)}^2 = \frac{1}{T}\int_0^T \|u_i^n(t) - u_i^m(t)\|^2 dt.$$

We have the estimate

$$\int_\Omega |u_i^n(t) - u_i^m(t)|^2 dx$$

$$= \frac{1}{T}\int_\Omega \int_0^T |u_i^n(t) - u_i^m(t)|^2 dt dx$$

$$+ 2\int_\Omega \int_s^t \left(\frac{du_i^n(\tau)}{dt} - \frac{du_i^m(\tau)}{dt}\right)(u_i^n(\tau) - u_i^m(\tau)) d\tau dx$$

$$\leq \frac{1}{T}\int_\Omega \int_0^T |u_i^n(t) - u_i^m(t)|^2 dt dx + 2\left\|\frac{du_i^n}{dt} - \frac{du_i^m}{dt}\right\|_{V_i^*} \|u_i^n - u_i^m\|_V.$$

Hence $\{u_i^n\}$ is a Cauchy sequence in $C([0, T]; L^2(\Omega))$. Thus, the sequence $\{u_i^n\}$ converges in $C([0, T]; L^2(\Omega))$ to a function $v_i \in C([0, T]; L^2(\Omega))$. Since $u_i^n(t) \to u_i(t) \in L^2(\Omega)$ for a.e. $t \in [0, T]$, we deduce that $u_i = v_i$ for a.e. $t \in [0, T]$. After redefining on a subset of zero-measure, we get $u_i^n \in C([0, T]; L^2(\Omega))$.

(ii) By compactness results, Proposition 2.1 [2], we get the following embeddings:

(i) $\mathcal{D}_0^{1, p_i}(\Omega, \sigma_i) \subset W_0^{1, \beta_i}$ continuously, if $1 \leq \beta_i < \dfrac{p_i N}{N + \alpha_i}$,

(ii) $\mathcal{D}_0^{1, p_i}(\Omega, \sigma_i) \subset L^{r_i}$ compactly, if $1 \leq r_i < \dfrac{p_i N}{N - p_i + \alpha_i}$.

We can choose $\gamma_i \in]\theta_i, p_i] \cap [p_i^*, q_i^*]$ such that $\mathcal{D}_0^{1, p_i}(\Omega, \sigma_i) \subset L^{\gamma_i}(\Omega)$. Since $\gamma_i^* \in [p_i, q_i]$, we have

$$L^{p_i}(\Omega) \cap L^{q_i}(\Omega) \cap L^2(\Omega) \subset L^{\gamma_i}(\Omega),$$

and therefore,

$$L^{\gamma_i}(\Omega) \subset L^{p_i^*}(\Omega) + L^{q_i^*}(\Omega) + L^2(\Omega).$$

So $\mathcal{D}_0^{1, p_i}(\Omega, \sigma_i) \subset L^{\gamma_i}(\Omega) \subset \mathcal{D}^{-1, p_i^*}(\Omega, \sigma_i) + L^{q_i^*}(\Omega) + L^2(\Omega).$

The boundedness of $\left\{\frac{du_i^n}{dt}\right\}$ in V_i ensures that $\left\{\frac{du_i^n}{dt}\right\}$ is also bounded in $L^s(0, T; W_i^*)$, where $s = \min(p_i^*, q_i^*, 2)$. By lemma in [18, p. 58], $\{u_i^n\}$ is precompact in $L^{p_i}(0, T; L^{\gamma_i}(\Omega))$ and therefore in $L^{\gamma_i}(0, T; L^{\gamma_i}(\Omega))$, this implies that $u_i^n \to u_i$ a.e. in Q_T.

2.2. Existence theorem

First, we specify our notion of weak solution.

Definition 2.1. A pair (u_1, u_2) is said to be a weak solution of $(\mathcal{S})$, if for $i = 1, 2$.

$$u_i \in V_i, \quad \frac{\partial u_i}{\partial t} \in V_i^*,$$

$$(u_1(., 0), u_2(., 0)) = (\varphi_1(.), \varphi_2(.)),$$

$$\int_\Omega \left(\frac{\partial u_i}{\partial t} w_i + \sigma_i |\nabla u_i|^{p_i - 2} \nabla u_i \nabla w_i + \left(-a_i |u_i|^{q_i - 2} u_i + f_i(x, u_1, u_2)\right) w_i \right) \times dxdt = 0,$$

for all test functions $w_i \in V_i$.

Theorem 2.1. *Under the assumptions* (H1)-(H3), *for each* $(\varphi_1, \varphi_2) \in L^2(\Omega) \times L^2(\Omega)$ *and* $T > 0$ *given, system* $(\mathcal{S})$ *has at least one weak solution on* $(0, T)$.

The main tools in the proof of this theorem are the Feado-Galerkin method and compactness arguments.

Let $\{e_j\}_{j=1}^{\infty}$ is a basis of $\mathcal{D}_0^{1, p_i}(\Omega, \sigma_i) \cap L^{q_i}(\Omega) \cap L^2(\Omega)$, which is orthogonal in $L^2(\Omega)$.

Proof. Consider the approximating solution $u_i^n(t)$ in the form

$$u_i^n(t) = \sum_{k=1}^{n} u_{ik}^n(t) e_k. \tag{2.1}$$

We get u_i^n from solving the system

$$\left\langle \frac{du_i^n}{dt}, e_k \right\rangle = -\langle \mathcal{A}_i u_i^n, e_k \rangle + \left\langle a_i |u_i^n|^{q_i - 2} u_i^n - f_i(x, u^n), e_k \right\rangle,$$

$$(u_i^n(0), e_k) = (\varphi_i, e_k), \quad k = 1, \dots, n.$$

It can be shown that the above system satisfies the Caracthodory's conditions; therefore, there exists solutions u_i^n in $[0, t_m], t_m < T$.

We now establish some a priori estimates for u_i^n and $\frac{du_i^n}{dt}$.

We have

$$\frac{1}{2} \frac{d}{dt} \sum_{i=1}^{2} \|u_i^n(t)\|_{L^2(\Omega)}^2 + \sum_{i=1}^{2} \int_{\Omega} \sigma_i |\nabla u_i^n|^{p_i} dx + \sum_{i=1}^{2} \int_{\Omega} a_i |u_i^n|^{q_i} dx$$

$$= \sum_{i=1}^{2} \int_{\Omega} (f_1(x, u^n) u_i^n) dx. \tag{2.2}$$

Using assumptions (H2) and (H3), we deduce that

$$\frac{1}{2} \frac{d}{dt} \sum_{i=1}^{2} \|u_i^n(t)\|_{L^2(\Omega)}^2 + \sum_{i=1}^{2} \int_{\Omega} \sigma_i |\nabla u_i^n|^{p_i} dx + d \sum_{i=1}^{2} \int_{\Omega} |u_i^n|^{q_i} dx$$

$$\leq M \sum_{i=1}^{2} \int_{\Omega} |u_i^n|^{p_i} dx + \int_{\Omega} h(x) dx. \tag{2.3}$$

Let $\theta_i = \max\left(\dfrac{p_i N}{N - p_i + \alpha_i}, 1\right)$.

Now, using the interpolation inequality, Young inequality, and the assumption $p_i < q_i$, we have

$$M\|u_i^n\|_{L^{p_i}(\Omega)}^{p_i} \le c\|u_i^n\|_{L^{q_i}(\Omega)}^{p_i\epsilon}\|u_i^n\|_{L^{r_i}(\Omega)}^{p_i(1-\epsilon)} \le c\|u_i^n\|_{L^{q_i}(\Omega)}^{p_i\epsilon}\|u_i^n\|_{\mathcal{D}_0^{1,p_i}(\Omega,\sigma_i)}^{p_i(1-\epsilon)},$$

for some $r_i \in (\theta_i, p_i)$ and $\epsilon \in (0, 1)$.

Hence

$$M\|u_i^n\|_{L^{p_i}(\Omega)}^{p_i} \le \frac{1}{2}\|u_i^n\|_{\mathcal{D}_0^{1,p_i}(\Omega,\sigma_i)}^{p_i} + c\|u_i^n\|_{L^{q_i}(\Omega)}^{q_i}$$

$$\le \frac{1}{2}\|u_i^n\|_{\mathcal{D}_0^{1,p_i}(\Omega,\sigma_i)}^{p_i} + \frac{d}{2}\|u_i^n\|_{L^{q_i}(\Omega)}^{q_i} + c'. \tag{2.4}$$

Substituting (2.4) in (2.3), we infer that

$$\sum_{i=1}^{2}\|u_i^n(t)\|_{L^2(\Omega)}^2 + \sum_{i=1}^{2}\int_\Omega \sigma_i|\nabla u_i^n|^{p_i}dx + d\sum_{i=1}^{2}\int_\Omega |u_i^n|^{q_i}dx$$

$$\le \sum_{i=1}^{2}\|u_i^n(0)\|_{L^2(\Omega)}^2 + 2t\left(c' + \int_\Omega h(x)dx\right), \tag{2.5}$$

for any $t \in (0, T)$.

The inequality (2.5) implies that

$\{u_i^n\}$ is bounded in $L^\infty(0, T; L^2(\Omega))$,

$\{u_i^n\}$ is bounded in $L^{p_i}(0, T; \mathcal{D}_0^{1,p_i}(\Omega,\sigma_i))$ and in $L^{p_i}(Q_T)$,

$\{u_i^n\}$ is bounded in $L^{q_i}(Q_T)$.

From Lemma 2.2, $u_i^n \to u_i$ a.e. in Q_T.

By Hölder inequality, we deduce that

$$|\langle \mathcal{A}_i u_i^n,\, v_i \rangle| = \left| \int_0^T dt \int_\Omega \sigma_i^{\frac{p_i - 1}{p_i}} |\nabla u_i^n|^{p_i - 2} \nabla u_i^n \left(\sigma_i^{\frac{1}{p_i}} \nabla v_i \right) \right|$$

$$\leq \|u_i^n\|_{L^{p_i}(0, T; \mathcal{D}_0^{1, p_i}(\Omega, \sigma_i))}^{\frac{p_i}{p_i^*}} \|v_i\|_{L^{p_i}(0, T; \mathcal{D}_0^{1, p_i}(\Omega, \sigma_i))}, \tag{2.6}$$

so, we infer that $\{\mathcal{A}_i u_i^n\}$ is bounded in $L^{p_i^*}(0,\, T;\, \mathcal{D}^{-1, p_i^*}(\Omega,\, \sigma_i))$, therefore,

$$\frac{du_i^n}{dt} \rightharpoonup \frac{du_i}{dt} \text{ in } V^*,$$

$$\mathcal{A}_i u_i^n \rightharpoonup \Psi_i \text{ in } L^{p_i^*}(0,\, T;\, \mathcal{D}^{-1, p_i^*}(\Omega,\, \sigma_i)).$$

On the other hand, we have the system

$$\frac{du_i^n}{dt} = -\mathcal{A}_i u_i^n - a_i |u_i^n|^{q_i - 1} u_i + f_i(x,\, u_1^n,\, u_2^n). \tag{2.7}$$

From the estimate (2.2), it follows

$$\sum_{i=1}^{2} \langle \mathcal{A}_i u_i^n,\, u_i^n \rangle = \int_0^T dt \int_\Omega \sigma_i |\nabla u_i^n|^{p_i} dx$$

$$= \int_0^T dt \int_\Omega \sum_{i=1}^{2} (-a_i |u_i^n|^{q_i} + f_i(x,\, u_1^n,\, u_2^n) u_i^n) dx$$

$$+ \frac{1}{2} \sum_{i=1}^{2} \left(\|u_i^n(0)\|_{L^2(\Omega)}^2 - \|u_i^n(T)\|_{L^2(\Omega)}^2 \right). \tag{2.8}$$

By the lower semi-continuity of $\|.\|_{L^2(\Omega)}$ and the Lebesgue dominated theorem, we obtain

$$\|u_i(T)\|^2_{L^2(\Omega)} \leq \liminf_{n\to\infty} \|u_i^n(T)\|^2_{L^2(\Omega)}, \tag{2.9}$$

and

$$\int_0^T dt \int_\Omega ((a_i|u_i|^{q_i} - f_i(x, u_1, u_2)u_i)dx)$$

$$= \lim_{n\to\infty} \int_0^T dt \int_\Omega (a_i|u_i^n|^{q_i} - f_i(x, u_1^n, u_2^n)u_i^n)dx. \tag{2.10}$$

Then

$$\overline{\lim_{n\to\infty}} \sum_{i=1}^{2} \langle \mathcal{A}_i u_i^n, u_i^n \rangle \leq \int_0^T dt \int_\Omega (a_i|u_i|^{q_i} - f_i(x, u_1, u_2)u_i)dx$$

$$+ \sum_{i=1}^{2} \frac{1}{2} \left(\|u_i(0)\|^2_{L^2(\Omega)} - \|u_i(T)\|^2_{L^2(\Omega)} \right). \tag{2.11}$$

In addition, from (2.7), we deduce

$$\frac{du_i}{dt} = \Psi_i - a_i|u_i|^{q_i-2}u_i + f_i(x, u_1, u_2) \text{ in } V_i^*. \tag{2.12}$$

Hence

$$\sum_{i=1}^{2} \langle \mathcal{A}_i u_i, u_i \rangle = \sum_{i=1}^{2} \langle a_i|u_i|^{q_i} u_i - f_i(x, u)u_i \rangle$$

$$+ \frac{1}{2} \sum_{i=1}^{2} \left(\|u_i(0)\|^2_{L^2(\Omega)} - \|u_i(T)\|^2_{L^2(\Omega)} \right). \tag{2.13}$$

From this estimate and (2.11), it follows that

$$\overline{\lim_{n\to\infty}} \sum_{i=1}^{2} \langle \mathcal{A}_i u_i^n, u_i^n \rangle \leq \sum_{i=1}^{2} \langle \Psi_i, u_i \rangle. \tag{2.14}$$

The property (3) of $\mathcal{A}_i$ implies $\mathcal{A}_i u_i = \Psi_i$ and we find that u_i is a weak solution of system $(\mathcal{S})$.

3. Global Attractor

For the convenience of the readers, we first use some concepts and results related to the theory of global attractors for multi-valued semiflows [21].

Definition 3.1. Let E be a Banach space. The mapping

$$\mathcal{G} : [0, +\infty[\times E \to E$$

is called an *m-semiflow*, if the following conditions are satisfied:

(i) $\mathcal{G}(0, w) = w$ for arbitrary $w \in E$;

(ii) $\mathcal{G}(t_1 + t_2, w) \subset \mathcal{G}(t_1, \mathcal{G}(t_2, w))$, for all $w \in E$, $t_1, t_2 \geq 0$.

Definition 3.2. The set $\mathcal{A}$ is said to be a global attractor of the m-semiflow $\mathcal{G}$, if the following conditions hold:

- $\mathcal{A}$ is attracting, i.e., $\operatorname{dist}(\mathcal{G}(t, B), \mathcal{A}) \to 0$ as $t \to \infty$, for all bounded subset $B \subset E$;
- $\mathcal{A}$ is negatively semi-invariant: $\mathcal{A} \subset \mathcal{G}(t, \mathcal{A})$ for arbitrary $t \geq 0$;
- If $\mathcal{B}$ is an attracting of $\mathcal{G}$, then $\mathcal{A} \subset \overline{\mathcal{B}}$.

Theorem 3.1 ([21]). *Suppose that the m-semiflow $\mathcal{G}$ has the following properties*:

(1) *$\mathcal{G}$ is pointwise dissipative*, *i.e.*, *there exists $K > 0$ such that for $u_0 \in E$, $u(t) \in \mathcal{G}(0, u_0)$ one has $\|u(t)\|_E \leq K$, if $t \geq t_0(\|u_0\|_E)$*;

(2) *$\mathcal{G}(t, .)$ is a closed map for any $t \geq 0$, i.e., if $\xi_n \to \xi$, $\eta_n \to \eta$, and $\xi_n \in \mathcal{G}(t, \eta_n)$, then $\xi \in \mathcal{G}(t, \eta)$*;

(3) *$\mathcal{G}$ is asymptotically upper semicompact*, *i.e.*, *if B is a bounded set in E such that for some $T(B)$, $\gamma^+_{T(B)}(B)$ is bounded*, *any sequence $\xi_n \in \mathcal{G}(t_n, B)$ with $t_n \to \infty$ is precompact in E. Here $\gamma^+_{T(B)}(B)$ is the orbit after the time $T(B)$.*

Then $\mathcal{G}$ has a compact global attractor in E. Moreover, if $\mathcal{G}$ is a strict m-semiflow, then $\mathcal{A}$ is invariant, i.e., $\mathcal{G}(t, \mathcal{A}) = \mathcal{A}$ for any $t \geq 0$.

By Theorem 2.1, we construct the multi-valued mapping as follows:

$$\mathcal{G}(t, (\varphi_1, \varphi_2)) = \left\{ \begin{matrix} u(t) = (u_1(t), u_2(t)) | \\ u(.) \text{ is the solution of } (\mathcal{S}), u_0 = u(0) = (\varphi_1, \varphi_2) \end{matrix} \right\}.$$

We now check that $\mathcal{G}$ is a strict m-semiflow in the sense of Definition 3.1. Assume that $\xi \in \mathcal{G}(t_1 + t_2, (\varphi_1, \varphi_2))$, then $\xi = u(t_1 + t_2)$, where $u(t)$ is a solution of system $(\mathcal{S})$. Denoting $v(t) = u(t_1 + t_2)$, we see that $v(.)$ is also in the set of solutions of system $(\mathcal{S})$ with respect to the initial condition $v(0) = u(t_2)$. Therefore, $\xi = v(t_1) \in \mathcal{G}(t_1, u(t_2)) \subset \mathcal{G}(t_2, \mathcal{G}(t_2, u_0))$. It remains to show that $\mathcal{G}(t_1, \mathcal{G}(t_2, u_0)) \subset \mathcal{G}(t_1 + t_2, u_0)$. If $\xi \in \mathcal{G}(t_1, \mathcal{G}(t_2, u_0))$, then $\xi = v(t_1)$, where $v(0) \in \mathcal{G}(t_2, u_0)$. One can suppose that $v(0) = u(t_2)$, where $u(0) = u_0$.

Set

$$w(\tau) = \begin{cases} u(\tau), & 0 \leq \tau \leq t_2, \\ u(\tau - t_2), & \tau \geq t_2. \end{cases}$$

Since u and v are the solutions of $(\mathcal{S})$, we obtain that w is a solution of $(\mathcal{S})$ with $w(0) = u(0) = u_0$. In addition, by the fact that $\xi = v(t_1) = w(t_1 + t_2)$, we have $\xi \in \mathcal{G}(t_1 + t_2, u_0)$.

In order to show the existence of a global attractor for the m-semiflow $\mathcal{G}$, we need the following proposition:

Proposition 3.1. *Assuming that* (H1)-(H3) *hold, then the m-semiflow $\mathcal{G}$ generated by $(\mathcal{S})$ is pointwise dissipative.*

Proof. Let $(u_1, u_2) \in \mathcal{G}(t, (\varphi_1, \varphi_2))$ and reasoning as in the proof of Theorem 2.1, we also have

$$\frac{d}{dt} \sum_{i=1}^{2} \|u_i(t)\|_{L^2(\Omega)}^2 + \sum_{i=1}^{2} \int_\Omega \sigma_i |\nabla u_1|^{p_i} dx + M \sum_{i=1}^{2} \|u_i(t)\|_{L^2(\Omega)}^{q_i} \leq c. \tag{3.1}$$

We deduce from Lemma 2.1 that $\mathcal{G}$ is pointwise dissipative.

Proposition 3.2. *Assuming that* (H1)-(H3) *hold, then the m-semiflow* $\mathcal{G}(t_0, .) : L^2(\Omega) \times L^2(\Omega) \to L^2(\Omega) \times L^2(\Omega)$ *is a compact mapping for each* $t_0 \in (0, T)$.

Proof. Assume that B is a bounded set in $L^2(\Omega) \times L^2(\Omega)$ and $\xi_n \in \mathcal{G}(t_0, B)$. By the definition of $\mathcal{G}$, there exists a sequence $\{u_i^n(t)\}$ such that $u_i^n(t)$ is the solution of $(\mathcal{S})$ with the initial data belongs to B and $u_i^n(t_0) = \xi_n$.

Then, we have

$$\frac{1}{2}\sum_{i=1}^{2}\|u_i^n(t)\|_{L^2(\Omega)}^2 + \sum_{i=1}^{2}\int_{Q_t}\sigma_i|\nabla u_i^n|^{p_i}\,dx + d\sum_{i=1}^{2}\int_{Q_t}|u_i^n|^{q_i}\,dx$$

$$= \frac{1}{2}\sum_{i=1}^{2}\|u_i^n(0)\|_{L^2(\Omega)}^2 + \sum_{i=1}^{2}\int_{Q_t} f_i(x, u^n)u_i^n, \tag{3.2}$$

for any $t \in (0, T)$.

By the same arguments as in proof of Theorem 3.1, we infer that

$u_i^n \to u_i$ a.e. in Q_T,

$u_i^n(t) \rightharpoonup u_i$ in $L^2(\Omega)$, for any $t \in [0, T]$,

$u_i^n \in V_i$ and $\dfrac{du_i^n}{dt} \in V_i^*$.

By Lemma 2.2, we obtain that u_i^n and u_i belong to $C([0, T]; L^2(\Omega))$. In the case $t = t_0$, one has $u_i^n(t_0) \rightharpoonup u_i$ in $L^2(\Omega)$.

We denote

$$J_n(t) = \sum_{i=1}^{2}\|u_i^n(t)\|_{L^2(\Omega)}^2 - ct\left(1 + \int_\Omega h(x)dx\right),$$

$$J(t) = \sum_{i=1}^{2} \|u_i(t)\|^2_{L^2(\Omega)} - ct\left(1 + \int_{\Omega} h(x)dx\right),$$

J_n and J are decreasing on $[0, T]$ for c chosen large enough. In addition, $J_n(t) \to J(t)$ for a.e. $t \in [0, T]$.

Suppose that $\{t_m\}$ is an increasing sequence in $[0, T]$, $t_m \to t_0$ as $m \to \infty$. Then

$$J_n(t_m) \to J_n(t_0) \text{ as } m \to \infty,$$

$$J_n(t_m) \to J(t_m) \text{ as } n \to \infty.$$

So

$$J_n(t_0) - J(t_0) \leq J_n(t_m) - J(t_0) = J_n(t_m) - J(t_m) + J(t_m) - J(t_0) < \varepsilon,$$

for $\varepsilon > 0$.

Similarly, $J(t_0) - J_n(t_0) < \varepsilon$. Therefore, $J_n(t_0) \to J(t_0)$ and then $\|u_i^n(t_0)\|_{L^2(\Omega)} \to \|u_i(t_0)\|_{L^2(\Omega)}$ as $n \to \infty$.

Theorem 3.2. *Assuming that* (H1)-(H3) *are satisfied*, *then the multi-valued semiflow* $\mathcal{G} : \mathbb{R}^+ \times (L^2(\Omega))^2 \mapsto \left(2^{L^2(\Omega)}\right)^2$ *associated with the boundary value problem* $(\mathcal{S})$ *possesses an invariant compact global attractor* $\mathcal{A}$ *in* $(L^2(\Omega))^2$.

Proof. Assume that $\xi_n \in \mathcal{G}(t, \eta_n)$, $\xi_n \to \xi$, and $\eta_n \to \eta$ in $L^2(\Omega)$. Then, there exists a sequence $\{u_i^n\}$ satisfying

$$u_i^n(t) = \xi_n, \quad u_i^n(0) = \eta_n.$$

It follows from the same arguments as in the proof of existence Theorem 2.1 that

$u_i^n(t) \to u_i(t)$ in $L^2(\Omega)$, for arbitrary $t \in [0, T]$ (and then $u_i(0) = \eta$),

$$\frac{du_i^n}{dt} \to \frac{du_i}{dt} \text{ in } V_i^*,$$

$$\mathcal{A}_i u_i^n \to \mathcal{A}_i u_i \text{ in } L^{p_i^*}(0, T; \mathcal{D}^{-1, p_i^*}(\Omega, \sigma_i)),$$

up to a subsequence. Hence, passing to the limit, the following equality in V_i^*.

$$\frac{du_i^n}{dt} + \mathcal{A}_i u_i^n + a_i |u_i^n|^{q_i - 2} u_i^n = f_i(x, u_1, u_2),$$

we conclude that $u_i^n(t)$ is the solution of $(\mathcal{S})$ with respect to initial condition $u_i(0) = \eta$. Thus, $\xi \in \mathcal{G}(t, \eta)$, one observes that

$$\mathcal{G}(t_n, B) = \mathcal{G}(t_0 + t_n - t_0, B) \subset \mathcal{G}(t_0, \mathcal{G}(t_n - t_0, B)) \subset \mathcal{G}(t_0, B_0),$$

where $t_0 > 0$ and B_0 is bounded set in $L^2(\Omega)$. Using Proposition 2.2, we see that, if $\xi_n \in \mathcal{G}(t_n, B))$, then $\{\xi_n\}$ is precompact in $L^2(\Omega)$.

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